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ON THE OPTIMAL CONTROL OF INTEGRAL-FUNCTIONAL EQUATIONS*

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The problem of the optimal control of stochastic integral-functional equations of neutral type with an intergral quality functional is considered. For the case of a linear quadratic problem an explicit form of the optimal control is presented.

A class of equations which originated in the synthesis of Volterra equations, and stochastic differential equations with after-effects of neutral type are discussed. The problem of the optimal control of such systems is an essential development of the theory of controlled differential equations /1—8/. Examples of real objects whose mathematical models contain equations with an after-effect are discussed in /9/. A study of integral equations of neutral type is essential in controlling the motion of bodies in a continuous medium, /10/. Volterra equations first arose in the theory of creep and form the basis of this theory /11, 12/.

1. Let $\{\xi_u(t), J(u), U\}$ be a certain problem of optimal control, with the trajectory of motion $\xi_u(t)$, the quality functional $J(u)$, and a set of feasible controls U. Also, let u_0 and u_ε be two elements from U, close to each other than $\varepsilon > 0$, and identical when $\varepsilon = 0$, for which the limit

$$J'(u_0) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [J(u_\varepsilon) - J(u_0)] \tag{1.1}$$

exists.

If u_0 is the optimal control of the problem $\{\xi_u(t), J(u), U\}$, that is $J(u_0) = \inf_{u \in U} J(u)$, the quantity $J'(u_0)$ is non-negative. Thus, the inequality $J'(u_0) \geqslant 0$ is a necessary condition for the optimality of the control u_0. In some cases it can be used to synthesize the optimal control.

The aim of the present paper is to calculate limit (1.1) for a problem of control with the trajectory of motion given by the stochastic integral-functional Eq.(1.2) and the quality functional (1.3)

$$\xi(t) = \eta(t) + \Phi(t, \theta_t\xi) + \int_0^t A(t, s, \theta_s\xi, u(s), ds) \tag{1.2}$$

$$\theta_0\xi = \varphi_0$$

$$A(t, s, \varphi, u, h) = a(t, s, \varphi, u)h + b(t, s, \varphi)(w(t+h) - w(t)) + \int c(z; t, s, \varphi)\nu^c([t, t+h], dz)$$

$$J(u) = M\left[F(\theta_T\xi) + \int_0^T G(s, \theta_s\xi, u(s))\,ds\right] \tag{1.3}$$

We will introduce the following notation and definitions: $\{\Omega, \sigma, P\}$ is the given probability space; $\{f_t\}$ is the stream of the σ-algebras $f_t \subset \sigma$, $t \in [0, T]$; $M_t = M\{\cdot / f_t\}$; $\|\varphi\|_0$ ($\|\varphi\|_1$) is the norm of the function $\varphi(t)$ defined by the relations

$$\|\varphi\|_0 = [\sup_{t \leqslant 0} M|\varphi(t)|^2]^{1/2}$$

$$(\|\varphi\|_1 = [\sup_{t \in [0, T]} M|\varphi(t)|^2]^{1/2})$$

H_0 (H_1) is the space of f_0 (f_t)-measurable functions $\varphi(t)$, $\varphi(t) \in R^n$, $t \in (-\infty, 0]$ ($[0, T]$) which are continuous from the right and bounded from the left, and are such that $\|\varphi\|_0 < \infty$ ($\|\varphi\|_1 < \infty$); U is the set of feasible controls, that is of f_t-measurable functions $u(t)$, $u(t) \in R^l$, $t \in [0, T]$ for which there exists a solution of (1.2), and the functional (1.3) is finite; U_0 is the set of f_t-measureable functions $u(t)$, $u(t) \in R^l$, $t \in [0, T]$ such that $\|u\|_1 < \infty$; $D(\alpha)$ is the set of f_t-measurable functions $\varphi(t)$ such that for certain $\alpha > 0$ and $c > 0$, the relation $M|\varphi(t) - \varphi(s)|^2 \leqslant c|t - s|^\alpha$ holds for any t and s from the definition domain of $\varphi(t)$; and S denotes a set of non-decreasing functions $K(\tau)$, $\tau = (-\infty, 0]$ which are continuous from the right, have a limit on the left, and are such that

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$$\int_{-\infty}^{0} dK(\tau) < \infty$$

We shall say that the function $K(\tau)$ from S has an isolated step at zero if there exists $\delta > 0$ such that in the segment $[-\delta, 0]$ it has a unique step at zero: $dK(0) = K(0) - K(-0)$; S_1 is the subset of the functions from S which have an isolated step at zero, less than unity, that is $dK(0) < 1$; S_0 is a subset of the functions from S which are continuous in a certain sufficiently small vicinity of zero $[-\delta, 0]$ (note that for $\tau < 0$ and $K_0(0) = K(-0)$ it follows from $K(\tau) \in S_1$ and $K_0(\tau) = K(\tau)$ that $K_0(\tau) \in S_0$); V, is the set of functions $R(t, \tau), t \in [0, T]$, negative and non-decreasing in $\tau \in [0, t]$ such that

$$\sup_{0 \leqslant t \leqslant T} \int_0^t dR(t, \tau) < \infty$$

and S_2 denotes the subset of the functions $K(\tau)$ from S for which the kernel $dK(\tau - t)$ has a resolvent in V.

If X and Y are two normal spaces, and $B(x)$ is a certain mapping of X and Y, then $\nabla B(x)$ is the Gateaux derivative of this mapping. For fixed $x_0 \in X$ $\nabla B(x_0)$ is a linear operator mapping X into Y (see /13/, p.471). For arbitrary x_0 and x_1 from X the relation

$$B(x_1) - B(x_0) = \int_0^1 \nabla B(x_0 + \tau(x_1 - x_0))(x_0 - x_1)\, d\tau \tag{1.4}$$

holds. If $Y = R^1$, then $\langle \nabla B(x_0), x \rangle$ is the value of the linear functional $\nabla B(x_0)$ on the element $x \in X$ (see /14/, p.62).

The letters c and α (with indices or without) denote various positive constants, $a \wedge b = \min[a, b]$. The scalar functions $F(\varphi)$, $G(t, \varphi, u)$, the n-dimensional functions $\Phi(t, \varphi)$, $a(t, s, \varphi, u)$, $c(z; t, s, \varphi)$, and the $n \times m$ matrix function $b(t, s, \varphi)$ are defined for $0 \leqslant s \leqslant t \leqslant T$, $z \in R^n$, $u \in R^l$, $\varphi \in H_0$. The centralized Poisson measure $\nu^\circ(t, A)$ with parameter $t\Pi(A)$ and the m-dimensional Wiener process $w(t)$ are mutually independent and f_t-measurable; $\eta(t)$ is an f_t-measurable random process, and θ_t is the family of shift operators: $\theta_t \xi(s) = \xi(t + s)$, $s \leqslant 0$, $t \geqslant 0$.

For $t < 0$, the process $\xi(t)$ is assumed to be known, and at the same time $\theta_0 \xi = \varphi_0 \in H_0$. For $t > 0$ it is determined by Eq.(1.2). It is proposed that the "splicing condition", characteristic for equations of neutral type (/9/, p.28).

$$\varphi_0(0) = \eta(0) + \Phi(0, \varphi_0) \tag{1.5}$$

is satisfied. Let

$$u_0 \in U$$
$$u_\varepsilon(t) = \begin{cases} v, & t \in [t_0 - \varepsilon, t_0), \quad 0 < \varepsilon < t_0 < T \\ u_0(t), & t \in [0, T] \setminus [t_0 - \varepsilon, t_0) \end{cases} \tag{1.6}$$
$$g_\varepsilon(t) = \frac{1}{\varepsilon}(\xi_\varepsilon(t) - \xi_0(t)), \quad p_\varepsilon(t) = \frac{1}{\varepsilon}(\Phi(t, \theta_t \xi_\varepsilon) - \Phi(t, \theta_t \xi_0))$$
$$C_\varepsilon(t, s, u, ds) = A(t, s, \theta_s \xi_\varepsilon, u, ds) - A(t, s, \theta_s \xi_0, u_0(s), ds)$$
$$\eta_\varepsilon(t) = \frac{1}{\varepsilon} \int_{t_0 - \varepsilon}^{t_0 \wedge t} C_\varepsilon(t, s, v, ds), \quad t \in [t_0 - \varepsilon, T]$$
$$\rho_\varepsilon(t) = \frac{1}{\varepsilon} \int_{t_0}^{t} C_\varepsilon(t, s, u_0(s), ds), \quad t \in [t_0, T]$$

where ξ_0 is the solution of Eq.(1.2) with the control u_0 and ξ_ε with the control u_ε.

Assuming that $\rho_\varepsilon(t) = 0$ when $t \in [0, t_0]$, and $\eta_\varepsilon(t) = 0$ when $t \in [0, t_0 - \varepsilon]$, we obtain

$$g_\varepsilon(t) = \eta_\varepsilon(t) + p_\varepsilon(t) - \rho_\varepsilon(t), \quad t \in [0, T]$$

Let

$$\lambda_\varepsilon^\tau(t) = \xi_0(t) - \tau(\xi_\varepsilon(t) - \xi_0(t)), \quad \tau \in [0, 1]$$
$$\Phi_\varepsilon(t) = \int_0^1 \nabla\Phi(t, \theta_t \lambda_\varepsilon^\tau)\, d\tau, \quad \varepsilon \geqslant 0$$
$$A_\varepsilon(t, s, h) = \int_0^1 \nabla A(t, s, \theta_s \lambda_\varepsilon^\tau, u_0(s), h)\, d\tau, \quad \varepsilon \geqslant 0$$

Then

$$p_\varepsilon(t) = \Phi_\varepsilon(t)\theta_t g_\varepsilon, \quad \rho_\varepsilon(t) = \int_{t_0}^{t} A_\varepsilon(t, s, ds)\theta_s g_\varepsilon$$

that is $g_\varepsilon(t)$ satisfies the equation

$$g_\varepsilon(t) = \eta_\varepsilon(t) + \Phi_\varepsilon(t)\theta_t g_\varepsilon + \int_{t_0}^{t} A_\varepsilon(t, s, ds)\theta_s g_\varepsilon \tag{1.7}$$

Also, consider the equations

$$g_0(t) = \eta_0(t) + \Phi_0(t)\theta_t g_0 + \int_{t_0}^{t} A_0(t, s, ds)\theta_s g_0 \tag{1.8}$$

$$\eta_0(t) = a(t, t_0, \theta_{t_0}\xi_0, v) - a(\cdot, u_0(t_0)) \tag{1.9}$$

Assuming that

$$\zeta_\varepsilon(t) = \Psi_\varepsilon(t)\theta_t g_0 + \int_{t_0}^{t} B_\varepsilon(t, s, ds)\theta_s g_0 \tag{1.10}$$

$$\Psi_\varepsilon(t) = \Phi_\varepsilon(t) - \Phi_0(t), \quad B_\varepsilon(t, s, h) = A_\varepsilon(t, s, h) - A_0(t, s, h)$$

$$l_\varepsilon(t) = g_\varepsilon(t) - g_0(t), \quad \gamma_\varepsilon(t) = \eta_\varepsilon(t) - \eta_0(t)$$

we obtain

$$l_\varepsilon(t) = \zeta_\varepsilon(t) + \gamma_\varepsilon(t) + \Phi_\varepsilon(t)\theta_t l_\varepsilon + \int_{t}^{t} A_\varepsilon(t, s, ds)\theta_s l_\varepsilon \tag{1.11}$$

Notice that the equation

$$\frac{1}{\varepsilon}(J(u_\varepsilon) - J(u_0)) = M\Big[\frac{1}{\varepsilon}(F(\theta_T\xi_\varepsilon) - F(\theta_T\xi_0)) + \frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0}(G(s, \theta_s\xi_\varepsilon, v) - G(s, \theta_s\xi_0, u_0(s)))\,ds + \frac{1}{\varepsilon}\int_{t_0}^{t}(G(s, \theta_s\xi_\varepsilon, u_0(s)) - G(s, \theta_s\xi_0, u_0(s)))\,ds\Big] \tag{1.12}$$

follows from (1.2), (1.3) and (1.6).
Let us introduce the following conditions:
1°. $\varphi \in H_0$; 2°. $\eta \in H_1$; 3°. $u_0 \in U_0$; 4°. $\varphi \in D(\alpha_1)$; 5°. $\eta \in D(\alpha_2)$; 6°. $u_0 \in D(\alpha_3)$; 7°. The random quantity v is $f_{t_0-\varepsilon}$-measurable, and $M|v|^2 < \infty$. The following notation is used in conditions 8°–14° :

$$Q_{ij}^k = \int_{-\infty}^{0} |\varphi_k(\tau)|^i\, dK_j(\tau), \quad Q_j = Q_{0j}^k, \quad K_2(\tau) = K_2(z, \tau)$$

$$P_{ij} = \int_{-\infty}^{0} |\varphi_1(\tau) - \varphi_2(\tau)|^i\, dK_j(\tau)$$

$$R_{ij} = \int_{-\infty}^{0} |\varphi_1(\tau) - \varphi_2(\tau)|^i\, |\varphi(\tau)|^i\, dK_j(\tau)$$

$$Z_j = \int_{-\infty}^{0} |\varphi_1(\tau)|\,|\varphi_2(\tau)|\, dK_j(\tau)$$

$$L_{ij}^k = (1 + |u_1|^2 + |u_2|^2)^k Q_j + Q_{ij}^1 + Q_{ij}^2$$

8°. The functions Φ, a, b and c are such that

$$|\Phi(t, \varphi_1)| \leqslant Q_0 + Q_{10}^1, \quad |a(t, s, \varphi_1, u)|^2 \leqslant (1 + |u|^2)Q_1 + Q_{21}^1$$

$$|b(t, s, \varphi_1)|^2 \leqslant Q_1 + Q_{21}^1, \quad |c(z; t, s, \varphi_1)|^2 \leqslant Q_2 + Q_{22}^1$$

9°. The functions Φ, a, b and c are such that

$$|\Phi(t_1, \varphi_1) - \Phi(t_2, \varphi_2)| \leqslant P_{10} + |t_1 - t_2|^{\alpha_4} L_{10}^c$$

$$|a(t_1, s_1, \varphi_1, u_1) - a(t_2, s_2, \varphi_2, u_2)|^2 \leqslant P_{21} + |u_1 - u_2|^2 Q_1 + (|t_1 - t_2|^{\alpha_4} + |s_1 - s_2|^{\alpha_4}) L_{21}^1$$

$$|b(t_1, s_1, \varphi_1) - b(t_2, s_2, \varphi_2)|^2 \leqslant P_{21} + |t_1 - t_2|^{\alpha_4} L_{21}^c$$

$$|c(z; t_1, s_1, \varphi_1) - c(z; t_2, s_2, \varphi_2)|^2 \leqslant P_{22} + |t_1 - t_2|^{\alpha_4} L_{22}^c$$

10°. The functions Φ, a, b and c have a Gateaux derivative with respect to φ, and at the same time for any φ_1, φ_2 from H_0,

$$|\nabla\Phi(t, \varphi_1)\varphi_2| \leqslant Q_{10}^2, \quad |\nabla c(z; t, s, \varphi_1)\varphi_2|^2 \leqslant Q_{22}^2$$
$$|\nabla a(t, s, \varphi_1, u)\varphi_2|^2 + |\nabla b(t, s, \varphi_1)\varphi_2|^2 \leqslant Q_{21}^2$$

11°. The functions Φ, a, b and c have a Gateaux derivative with respect to φ, and at the same time for any φ_1, φ_2 and φ from H_0,

$$\begin{aligned}&|(\nabla\Phi(t, \varphi_1) - \nabla\Phi(t, \varphi_2))\varphi|^2 + \\ &\quad |(\nabla a(t, s, \varphi_1, u) - \nabla a(t, s, \varphi_2, u))\varphi|^2 + \\ &\quad |(\nabla b(t, s, \varphi_1) - \nabla b(t, s, \varphi_2))\varphi|^2 \leqslant R_{21} \\ &|(\nabla c(z; t, s, \varphi_1) - \nabla c(z; t, s, \varphi_2))\varphi|^2 \leqslant R_{22}\end{aligned}$$

12°. The functions F and G are such that

$$|F(\varphi_1)| \leqslant Q_1 + Q_{21}^1$$
$$|G(t, \varphi_1, u)| \leqslant (1 + |u|^2) Q_1 + Q_{21}^1$$

13°. The function G is such that

$$|G(t_1, \varphi_1, u_1) - G(t_2, \varphi_2, u_2)| \leqslant (L_{21}^1)^{1/2}[|u_1 - u_2| + P_{11}] + |t_1 - t_2|^{\alpha} L_{21}^1$$

14°. The functions F and G have a Gateaux derivative with respect to φ, and at the same time for φ_1, φ_2 and φ from H_0,

$$|\langle\nabla F(\varphi_1), \varphi_2\rangle| \leqslant Q_1 - Z_1$$
$$|\langle\nabla G(t, \varphi_1, u), \varphi_2\rangle| \leqslant (1 + |u|) Q_1 - Z_1$$
$$|\langle\nabla F(\varphi_1) - \nabla F(\varphi_2), \varphi\rangle| + |\langle\nabla G(t, \varphi_1, u) - \nabla G(t, \varphi_2, u), \varphi\rangle| \leqslant R_{11}$$

It is assumed in Conditions 8°–14° that

$$K_0 \in S_1 \cap S_2, \quad K = K_1 + \int K_2(z)\,\Pi(dz) \in S$$

We shall assume that the functions K_0, K_1, K_2 are the same for all conditions.

Theorem 1. Let Conditions 1°–14° be satisfied. Then for any $t_0 \in [0, T]$ the limit (1.1), (1.6) for the control problem (1.2), (1.3) exists, and is

$$J'(u_0) = M\Big[G(t_0, \theta_{t_0}\xi_0, v) - G(\cdot, u_0(t_0)) + \langle\nabla F(\theta_T\xi_0), \theta_T\gamma_0\rangle + \int_{t_0}^{T}\langle\nabla G(s, \theta_s\xi_0, u_0(s)), \theta_s\gamma_0\rangle\, ds\Big] \tag{1.13}$$

where $\gamma_0(t)$ is the solution of Eq. (1.8).

2. The following assertions are necessary to prove Theorem 1.

Lemma 1. Let $\alpha(t)$ be a non-negative function which satisfies the inequality

$$\alpha(t) \leqslant \beta(t) + \int_{-t}^{0}\alpha(t + s)\, dK(s), \quad t \in [0, T], \quad K \in S_0 \cap S_2$$

where $\beta(t)$ is a non-negative, non-decreasing and continuously differentiable function. Then $\alpha(s) \leqslant c\beta(t)$.

Proof. We introduce a sequence of functions $\gamma_n(t)$ such that

$$\gamma_0(t) = \alpha(t), \quad \gamma_n(t) = \beta(t) + \int_{-t}^{0}\gamma_{n-1}(t + s)\, dK(s), \quad n = 1, 2, \ldots$$

It can be shown that $\gamma_n(t) \geqslant \gamma_{n-1}(t)$ for all $n = 1, 2, \ldots$ and all $t \in [0, T]$. Let $\alpha_0(t)$ be the solution of the equation

$$\alpha_0(t) = \beta(t) + \int_{-t}^{0}\alpha_0(t + s)\, dK(s)$$

This solution exists and is unique (see /9/, p.30). Thus $\lim \gamma_n(t) = \alpha_0(t)$ as $n \to \infty$, with $\alpha(t) \leqslant \alpha_0(t)$. We have

$$\alpha_0(t) = \beta(t) + \int_0^t \alpha_0(s)\, dK(s - t) = \beta(t) + \int_0^t dR(t, s)\,\beta(s) \leqslant \beta(t)\Big[1 + \int_0^t dR(t, s)\Big] \leqslant c\beta(t)$$

Theorem 2. Let $u \in U_0$, and let Conditions 1°, 2°, 8° and 9° be satisfied. Then a unique solution of Eq. (1.2) exists in H_1.

Proof. Assume that

$$\theta_0\xi_n = \varphi_0, \quad n \geqslant 0, \quad \xi_0(t) = \eta(t) \tag{2.1}$$
$$\xi_{n+1}(t) = \eta(t) + \Phi(t, \theta_t\xi_{n+1}) + \int_0^t A(t, s, \theta_s\xi_n, u(s), ds)$$
$$y_n(t) = \mathbf{M}|\xi_n(t)|^2, \quad z_n(t) = \sup_{0 \leqslant s \leqslant t} y_n(s)$$

The function $z_n(t)$ is uniformly bounded. In fact, it follows from Condition 8° that

$$|\xi_{n+1}(t)|(1 - dK_0(0)) \leqslant c + |\eta(t)| + \int_{-\infty}^{-0} |\xi_{n+1}(t+s)|\, dK_0(s) + \left|\int_0^t A(t, s, \theta_s\xi_n, u(s), ds)\right| \tag{2.2}$$

From Conditions 1°, 2°, and 8°, the properties of stochastic integrals (see /15/, p.138) and Lemma 1, we can obtain the estimate

$$z_{n+1}(t) \leqslant c\left[1 + \int_0^t z_n(s)\, ds\right] \leqslant ce^{cT} + \|\eta\|_1 \frac{(cT)^{n+1}}{(n+1)!} \tag{2.3}$$

whence follows the uniform boundedness of $z_n(t)$.

Now let $z_n(t) = \sup_{0 \leqslant s \leqslant t} M|\xi_n(s) - \xi_{n-1}(s)|^2$. Using Condition 9°, similarly to (2.3) we obtain $z_{n+1}(t) \leqslant z_1(T)(cT)^n/n!$, $z_1(T) < \infty$, and $\lim z_n(t) = 0$ as $n \to \infty$ uniformly in $t \in [0, T]$. Consequently, $\xi_n(t)$ converges in the mean square to a certain process $\xi(t)$ which is a unique solution of Eq. (1.2), with $\|\xi\|_1 < \infty$ (see /15/, p. (238).

Notice that if Conditions 3° and 7° are satisfied, the control u_ε belongs to U_0.

Corollary 1. Let conditions 1°, 2°, 8°, 9° and 12° be satisfied. Then an arbitrary control from U_0 is feasible, i.e. $U_0 \subset U$.

Corollary 2. Let Conditions 1°–3°, 8° and 9° be satisfied. Then there exists in H_1 a unique solution of Eq. (1.2) for the control u_0. If additionally, Condition 7° is satisfied, then there exists in H a unqiue solution of Eq. (1.2) for the control u_ε. If in addition Condition 10° is satisfied, then unique solutions of Eqs. (1.7), (1.8) and (1.11) exist in H_1.

Theorem 3. Let the condition (1.5) and Conditions 1°–6°, 8° and 9° be satisfied. Then

$$\xi_0 \in D(\alpha), \quad \alpha = \min\{1, \alpha_1, \alpha_2, 2\alpha_4, \alpha_5, \alpha_7, \alpha_8\}$$

Proof. The existence of ξ_0 follows from Corollary 2. The inequality

$$M|\xi_0(t_1) - \xi_0(t_2)|^2 \leqslant c|t_1 - t_2|^\alpha, \quad \forall t_1, t_2 \in [0, T]$$

is proved in two stages. First, let $t_2 = 0, t_1 = t, z(t) = M|\xi_0(t) - \varphi_0(0)|^2$. An estimate of $|\xi_0(t) - \varphi_0(0)|$ analogous to (2.2) is obtained from (1.2) and (1.5). Then using Conditions 1°, 3°–5°, 8°, 9°, and $\xi_0 \in H_1$, and the relations

$$M|\xi_0(t+\tau) - \varphi_0(\tau)|^2 \leqslant 2[z(t+\tau) + |\tau|^{\alpha_1}]$$

we derive the inequality

$$z(t) \leqslant c\left[t^\alpha + \int_{-t}^{-0} z(t+\tau)\, dK_0(\tau)\right]$$

Hence (see Lemma 1) $z(t) \leqslant ct^\alpha$. Now assuming that

$$t_2 = t < t_1 = t + \Delta, \quad z(t) = M|\xi_0(t+\Delta) - \xi_0(t)|^2$$

making use of the similar previous estimate we finally have $z(t) \leqslant c\Delta^\alpha$. The theorem is proved.

Lemma 2. Let Conditions 1°-9° be satisfied. Then uniformly in $t \in [t_0, T]$ we have $\lim_{\varepsilon \to 0} M|\gamma_\varepsilon(t)|^2 = 0$.

Proof. Let us write $\gamma_\varepsilon(t)$ in the form

$$\gamma_\varepsilon(t) = \frac{1}{\varepsilon}\sum_{i=1}^{6} \delta_i(t)$$
$$\delta_1(t) = \int_{t_0-\varepsilon}^{t_0} [a(t, s, \theta_s\xi_\varepsilon, v) - a(t, s, \theta_s\xi_0, v)]\, ds$$

$$\delta_2(t) = \int_{t_0-\varepsilon}^{t_0} [a(t, s, \theta_s\xi_0, v) - a(t, t_0, \theta_{t_0}\xi_0, v)]\, ds$$

$$\delta_3(t) = \int_{t_0-\varepsilon}^{t_0} [a(t, t_0, \theta_{t_0}\xi_0, u_0(t_0)) - a(t, s, \theta_s\xi_0, u_0(s))]\, ds$$

$$\delta_4(t) = \int_{t_0-\varepsilon}^{t_0} [b(t, s, \theta_s\xi_\varepsilon) - b(t, s, \theta_s\xi_0)]\, dw(s)$$

$$\delta_5(t) = \int_{t_0-\varepsilon}^{t_0} \int [c(z; t, s, \theta_s\xi_\varepsilon) - c(z; t, s, \theta_s\xi_0)]\, \nu^0(ds, dz)$$

As in /16/, the estimates

$$M|\delta_1(t)|^2 \leqslant c\varepsilon^4, \quad M|\delta_2(t)|^2 \leqslant c\varepsilon^2(\varepsilon^{\alpha} + \varepsilon^{\alpha_4}), \quad M|\delta_3(t)|^2 \leqslant c\varepsilon^2(\varepsilon^{\alpha} + \varepsilon^{\alpha_3} + \varepsilon^{\alpha_4}), \quad M|\delta_4(t)|^2 + M|\delta_5(t)|^2 \leqslant c\varepsilon^3$$

follow from Condition 9° and Corollary 2. This proves the lemma.

Lemma 3. Let Conditions 1°–3°, 7°–11° be satisfied. Then, uniformly in $t \in [t_0, T]$ we have $\lim_{\varepsilon\to 0} M|\zeta_\varepsilon(t)|^2 = 0$.

Proof. By (1.10),

$$M|\zeta_\varepsilon(t)|^2 \leqslant c \sum_{i=1}^{4} \delta_i(t)$$

$$\delta_1(t) = \int_0^1 M|[\nabla\Phi(t, \theta_t\lambda_\varepsilon^\tau) - \nabla\Phi(t, \theta_t\xi_0)]\,\theta_t q_0|^2\, d\tau$$

$$\delta_2(t) = \int_{t_0}^{t}\int_0^1 M|[\nabla a(t, s, \theta_s\lambda_\varepsilon^\tau, u_0(s)) - \nabla a(\cdot, \theta_s\xi_0, \cdot)]\,\theta_s q_0|^2\, d\tau\, ds$$

$$\delta_3(t) = \int_{t_0}^{t}\int_0^1 M|[\nabla b(t, s, \theta_s\lambda_\varepsilon^\tau) - \nabla b(\cdot, \theta_s\xi_0)]\,\theta_s q_0|^2\, d\tau\, ds$$

$$\delta_4(t) = \int_{t_0}^{t}\int\int_0^1 M|[\nabla c(z; t, s, \theta_s\lambda_\varepsilon^\tau) - \nabla c(\cdot, \theta_s\xi_0)]\,\theta_s q_0|^2\, d\tau\,\Pi(dz)\, ds$$

Let $\chi_N(s)$ be an indicator of the set $\{\omega: q_0(s) > N\}$. By expressing $q_0(s)$ in the form $q_0(s) = q_0(s)\chi_N(s) + q_0(s)(1 - \chi_N(s))$ and making use of Conditions 10° and 11°, we can show that

$$\delta_1(t) < c[\|q_0\chi_N\|^2 + \varepsilon^2 N^2\|q_\varepsilon\|^2]$$

For any $\delta > 0$ an N exists such that $\|q_0\chi_N\|^2 < \delta/(2c)$. Let us fix N and select ε so that $\varepsilon^2 N^2\|q_\varepsilon\|^2 < \delta/(2c)$. Then $\delta_1(t) < \delta$. Similar estimate hold for $\delta_i(t)$, $i = 2, 3, 4$, as well. Thus, for any $\delta > 0$ we can find $\varepsilon > 0$ such that $M|\zeta_\varepsilon(t)|^2 < \delta$. The lemma is proved.

Corollary 3. Let Conditions 1°–11° be satisfied. Then $\lim_{\varepsilon\to 0} M|l_\varepsilon(t)|^2 = 0$ uniformly in $t \in [t_0, T]$.

Proof. For any $\delta > 0$ we can find $\varepsilon > 0$ such that $M|\zeta_\varepsilon(t)|^2 + M|\gamma_\varepsilon(t)|^2 < \delta$ (see Lemma 2 and 3). Assuming $z_\varepsilon(t) = \sup_{0\leqslant s\leqslant t} M|l_\varepsilon(s)|^2$, similarly to (2.3) we obtain

$$z_\varepsilon(t) \leqslant c\left[\delta + \int_{t_0}^{t} z_\varepsilon(s)\, ds\right]$$

Hence, by the Gronwall - Bellman lemma, we obtain the necessary proof.

Lemma 4. Let Conditions 1°–10°, 13° be satisfied, and let

$$g_\varepsilon(s) = M[G(s, \theta_s\xi_\varepsilon, v) - G(s, \theta_s\xi_0, u_0(s))]$$

$$\mu_\varepsilon = \frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} g_\varepsilon(s)\, ds$$

Then $\mu = \lim_{\varepsilon\to 0}\mu_\varepsilon = g_0(t_0)$.

Proof. Suppose that

$$\delta_\varepsilon = \frac{1}{\varepsilon} M\int_{t_0-\varepsilon}^{t_0} [G(s, \theta_s\xi_\varepsilon, v) - G(t_0, \theta_{t_0}\xi_0, v)]\, ds + \frac{1}{\varepsilon} M\int_{t_0-\varepsilon}^{t_0} [G(t_0, \theta_{t_0}\xi_0, u_0(t_0)) - G(s, \theta_s\xi_0, u_0(s))]\, ds$$

Then $\mu_\varepsilon = \mu + \delta_\varepsilon$. Using Conditions 13°, 3°, 6°, 7°, Corollary 2 and Theorem 3, we can show (see /16/) that

$$|\delta_\varepsilon| \leqslant c\,[\sqrt{\varepsilon^2+\varepsilon^{\alpha}} + \varepsilon^{\alpha_0} + \varepsilon^{\alpha_0/2} + \varepsilon^{\alpha/2}]$$

i.e. $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$. The lemma is proved.

Lemma 5. Let Conditions 1°–11°, 14° be satisfied, and let

$$\beta_\varepsilon = \frac{1}{\varepsilon} M\,[F(\theta_T\xi_\varepsilon) - F(\theta_T\xi_0)],\quad \beta = M\,\langle \nabla F(\theta_T\xi_0), \theta_T q_0\rangle$$

Then $\lim_{\varepsilon\to 0}\beta_\varepsilon = \beta$, and in addition

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon} M\int_{t_0}^{T}[G(s,\theta_s\xi_\varepsilon,u_0(s)) - G(s,\theta_s\xi_0,u_0(s))]\,ds = M\int_{t_0}^{T}\langle \nabla G(s,\theta_s\xi_0,u_0(s)), \theta_s q_0\rangle\,ds$$

Proof. Suppose that

$$\delta_\varepsilon = M\,\langle \nabla F(\theta_T\xi_0),\ \theta_T l_\varepsilon\rangle + M\int_0^1 \langle \nabla F(\theta_T\lambda_\varepsilon^\tau) - \nabla F(\theta_T\xi_0), \theta_T q_\varepsilon\rangle\,d\tau$$

Then $\beta_\varepsilon = \beta + \delta_\varepsilon$. Using Condition 14° and Corollaries 2 and 3, we can show that $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$ (see /16/). The proof of the second assertion is similar. The lemma is proved.

Now the proof of Theorem 1 follows from (1.12) and Lemmas 4 and 5.

3. We shall demonstrate the possibility of a synthesis of the optimal control with the help of the condition $J'(u_0) \geqslant 0$, using as an example the following problem of controlling linear equations with a quadratic quality functional

$$\xi(t) = \eta(t) + \int_{-\infty}^{0} dK(t,s)\,\xi(t+s) + \int_0^t a(t,s)\,u(s)\,ds \tag{3.1}$$

$$J(u) = M\left[\xi^*(T)\,H\xi(T) + \int_0^T u^*(s)\,N(s)\,u(s)\,ds\right] \tag{3.2}$$

Here $\eta(t)$ is the random process satisfying Conditions 2° and 5°; $a(t,s)$ is a non-random, bounded $n\times l$ matrix, Hölderian with respect to both variables; $N(s)$ is a non-random, Hölder, bounded and positive $l\times l$ matrix; H is a non-random, non-negative $n\times n$ matrix; and $K(t,s)$ is a non-random $n\times n$ matrix such that

$$\sup_{0\leqslant t\leqslant T}|dK(t,s)| \leqslant dK_0(s),\quad K_0 \in S_1\cap S_2$$
$$|dK(t_1,s) - dK(t_2,s)| \leqslant |t_1-t_2|^{\alpha}\,dK_0(s)$$

Suppose that $dR(t,\tau)$ is the resolvent of the kernel $dK(t,\tau-t)$. We assume that

$$\psi(t,t_0,f(\cdot,s)) = f(t,s) + \int_{t_0}^{t} dR(t,\tau)\,f(\tau,s)$$

for an arbitrary matrix $f(\tau,s)$. Then

$$q_0(t) = \psi(t,t_0,a(\cdot,t_0))\,(v-u_0(t_0)),\quad t\in[t_0,T]$$

Let us write $J'(u_0)$ in the form

$$J'(u_0) = M\,[(v-u_0(t_0))^*\,N(t_0)\,(v-u_0(t_0)) +$$
$$2\,(v-u_0(t_0))^*\,(N(t_0)\,u_0(t_0) + \psi^*(T,t_0,a(\cdot,t_0))\,HM_{t_0}\xi_0(T))]$$

For $J'(u_0)$ to be non-negative it is necessary and sufficient that the optimal control of problem (3.1), (3.2) should have the form

$$u_0(t_0) = -N^{-1}(t_0)\,\psi^*(T,t_0,a(\cdot,t_0))\,HM_{t_0}\xi_0(T)$$

Computing $M_{t_0}\xi_0(T)$ from (3.1), the control $u_0(t_0)$ can be converted to the form

$$u_0(t_0) = p(t_0)\left[\zeta_0(t_0) + \int_0^{t_0}\psi(T,t_0,a_{t_0}(\cdot,s))\,u_0(s)\,ds\right]$$

$$p(t_0) = -N^{-1}(t_0)\,\psi^*(T,t_0,a(\cdot,t_0))\,H\,[E +$$
$$\int_{t_0}^{T}\psi(T,s,a(\cdot,s))\,N^{-1}(s)\,\psi^*(T,s,a(\cdot,s))\,ds\,H\Big]^{-1}$$

$$\zeta_0(t_0) = \psi(T,t_0,b(\cdot,t_0)) + \psi(T,t_0,E)\,\xi_0(t_0) +$$

$$\int_0^{t_0} \psi(T, t_0, dK_{t_0}(\cdot, s))\xi_0(s), \quad b(t,s) = M_s(\eta(t) - \eta(s))$$

$$a_{t_0}(t,s) = a(t,s) - a(t_0,s), \quad dK_{t_0}(t,s) = dK(t, s-t) - dK(t_0, s-t_0)$$

Let $Q(t,s)$ be a resolvent of the kernel $p(t)\psi(T,t,a_t(\cdot,s))$. Then for any arbitrary $t \in [0, T]$ the optimal control takes the form

$$u_0(t) = p(t)\zeta_0(t) + \int_0^t Q(t,s)p(s)\zeta_0(s)\,ds$$

On subsitituting $\zeta_0(t)$ into the above we obtain

$$u_0(t) = \alpha(t) + p(t)\psi(T,t,E)\xi_0(t) + \int_0^t dR_0(t,\tau)\xi_0(\tau) \tag{3.3}$$

$$\alpha(t) = p(t)\psi(T,t,b(\cdot,t)) + \int_0^t Q(t,s)p(s)\psi(T,s,b(\cdot,s))\,ds$$

$$dR_0(t,\tau) = p(t)\psi(T,t,dK_t(\cdot,\tau)) + Q(t,\tau)p(\tau)\psi(T,\tau,E)\,d\tau + \int_\tau^t Q(t,s)p(s)\psi(T,s,dK_s(\cdot,\tau))\,ds$$

Clearly, the control u_0 obtained, as a feedback, is feasible. Here the proof that the solution of (3.1) exists and is unique is analogous to that of Theorem 2.

Using the methods developed in /17–19/ we can demonstrate that the control (3.3) is ε-optimal for the problem of controlling a quasilinear integral equation which differs little from Eq.(3.1).

4. *Example 1.* The controlled motion of aircraft is described (see /10/) by systems of linear integro-differential equatons of the form

$$\xi^{\cdot}(t) = A_0(t)\xi(t) + \int_0^t A_1(t-s)\xi^{\cdot}(s)\,ds + \int_0^t A_2(t-s)\xi(s)\,ds + a(t)u(t) + \sigma(t)w^{\cdot}(t) \tag{4.1}$$

As mentioned in /10/, the creation of effective methods for optimal control by such systems "still remains an unsolved problem".

Let us show that Eq.(4.1) reduces to the form (3.1) and therefore the solution of the control problem (4.1), (3.2) can be obtained as a special case of problem (3.1), (3.2).

In fact, on integrating (4.1), we obtain

$$\xi(t) = \eta(t) + \int_0^t K(t,s)\xi(s)\,ds + \int_0^t a(s)u(s)\,ds$$

$$\eta(t) = (E - B_1(t))\xi(0) + \int_0^t \sigma(s)\,dw(s), \quad K(t,s) = A_0(s) + A_1(t-s) + B_2(t-s), \quad B_i(t) = \int_0^t A_i(s)\,ds, \quad i = 1, 2$$

Let $R(t,s)$ be the resolvent of the kernel $K(t,s)$, and

$$B_0(t) = E + \int_t^T R(T,s)\,ds, \quad p(t) = -N^{-1}(t)a^*(t)B_0^*(t)H\left[E + \int_t^T B_0(s)a(s)N^{-1}(s)a^*(s)B_0^*(s)\,dsH\right]^{-1}$$

$$R_0(t,s) = K(T,s) + \int_t^T R(T,\tau)K(\tau,s)\,d\tau - B_0(t)K(t,s)$$

Then the optimal control of problem (4.1), (3.2) is

$$u_0(t) = p(t)\left[B_0(t)\xi_0(t) + \int_0^t R_0(t,s)\xi_0(s)\,ds - \int_t^T B_0(s)A_1(s)\,ds\,\xi_0(0)\right]$$

Example 2. Consider the following problem of the optimal control of the stochastic differential equation of neutral type

$$\xi^{\cdot}(t) = b\xi^{\cdot}(t-h) + au(t) + w^{\cdot}(t), \ t \in [0, T]; \ a \neq 0, \ h \in (0, T)$$

$$J(u) = M\left[\xi^2(T) + \int_0^T u^2(s)\,ds\right]$$

in which $\xi(t)=0$ when $t\leqslant 0$, and $w(t)$ is a Wiener process.

Here

$$dK(t,\tau)=bb(h+\tau)\,d\tau,\quad dR(t,\tau)=\sum_{i=1}^{\infty} b^i\delta(ih+\tau-t)\,d\tau$$

Suppose that $(T-t)/h$ is non-integer, $n(t)=[(T-t)/h]+1$, $m(t,b)=n(t)$ for $b=1$, and $m(t,b)=(1-b^{n(t)})/(1-b)$ for $b\neq 1$. Then the optimal control has the form

$$u_0(t)=-m(t,b)\,[m(t,b)(\xi_0(t)-b\xi_0(t-h))+b^{n(t)}\xi_0(T-n(t)h)]\times \left[\frac{1}{a}+a\int_t^T m^2(s,b)\,ds\right]^{-1}$$

for almost all $t\in[0,T]$ (with the exception of $t_i=T-ih$, $i=1,\ldots,[T/h]$).

Note that the necessary condition of optimality for stochastic integral-functional equations was also given in /19/. Earlier it was obtained for stochastic differential equations (ordinary and partial) in /20–23/, and for stochastic Volterra equations in /24/.

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